

# Optimal bounds for periodic mixtures of ferromagnetic interactions

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## 1 Introduction

In this paper we give optimal bounds for the homogenization of periodic Ising systems of the form

$$\sum_{ij} c_{ij} (u_i - u_j)^2$$

where  $u_i \in \{-1, 1\}$ , the sum runs over all nearest neighbours in a square lattice, and the bonds  $c_{ij}$  may take two positive values  $\alpha$  and  $\beta$  with

$$\alpha < \beta.$$

Such bounds are given in terms of the *volume fraction* (proportion)  $\theta$  of  $\beta$ -bonds as follows. To each such system we associate a *homogenized surface tension*  $\varphi$ . We show that all possible such  $\varphi$  are the (positively homogeneous of degree one) convex functions such that

$$\alpha(|\nu_1| + |\nu_2|) \leq \varphi(\nu) \leq c_1|\nu_1| + c_2|\nu_2|, \quad (1)$$

where the coefficients  $c_1$  and  $c_2$  satisfy

$$c_1 \leq \beta, \quad c_2 \leq \beta, \quad c_1 + c_2 = 2(\theta\beta + (1 - \theta)\alpha). \quad (2)$$

The continuous counterpart of this problem is the determination of optimal bounds for Finsler metrics obtained from the homogenization of periodic Riemannian metrics (see [1, 7, 6]) of the form

$$\int_a^b a(u(t)) |u'|^2 dt,$$

and  $a(u)$  is a periodic function in  $\mathbb{R}^2$  taking only the values  $\alpha$  and  $\beta$ . This problem has been studied in [11], where it is shown that homogenized metrics satisfy

$$\alpha \leq \varphi(\nu) \leq (\theta\beta + (1 - \theta)\alpha),$$

but the optimality of such bounds is not proved. A ‘dual’ equivalent formulation in dimension 2 is obtained by considering the homogenization of periodic perimeter functionals of the form

$$\int_{\partial A} a(x) d\mathcal{H}^1(x)$$

with the same type of  $a$  as above (see [3, 4]). The corresponding  $\varphi$  in this case can be interpreted as the homogenized surface tension of the homogenized perimeter functional.

The discrete setting allows to give a (relatively) easy description of the optimal bounds in a way similar to the treatment of mixtures of linearly elastic discrete structures [8]. The bounds obtained by sections and by averages in the elastic case have as counterpart *bounds by projection*, where the homogenized surface tension is estimated from below by considering the minimal value of the coefficient on each section, and *bounds by averaging*, where coefficients on a section are substituted with their average. The discrete setting allows to construct (almost-)optimal periodic geometries, which optimize one type or the other of bound in each direction. We shortly describe the ‘extreme’ geometries in Fig. 1 and Fig. 2, where  $\alpha$ -connections are represented as dotted lines,  $\beta$ -connections are represented as solid lines, and the nodes with the value  $+1$  or  $-1$  as white circles or black circles, respectively. In Fig. 1 there are pictured the periodicity cell of a mixture giving as a result the lower bound  $\alpha(|\nu_1| + |\nu_2|)$  and an interface with minimal energy. Fig. 2 represents the

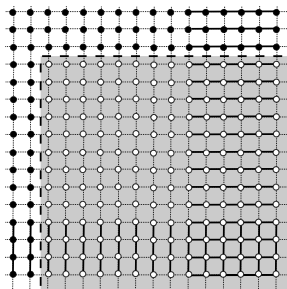


Figure 1: periodicity cell for a mixture giving the lower bound

periodicity cell of a mixture giving an upper bound of the form  $c_1|\nu_1| + c_2|\nu_2|$ . Note that the interface pictured in that figure crosses exactly a number of bonds proportional to the percentage  $\theta_v$  of  $\beta$ -bonds in the horizontal direction.

It must be noted that, contrary to the elastic case, the bounds (i.e., the sets of possible  $\varphi$ ) are increasing with  $\theta$ , and in particular they always contain the minimal surface tension  $\alpha(|\nu_1| + |\nu_2|)$ .

We can picture the bounds in terms of their Wulff shape; i.e., the solutions  $A_\varphi$  centered in 0 to the problem

$$\max\left\{|A| : \int_{\partial A} \varphi(\nu) d\mathcal{H}^1(x) = 1\right\}.$$

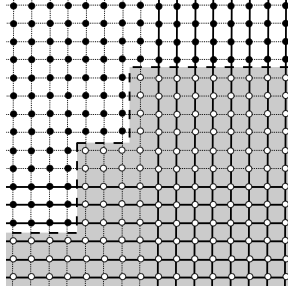


Figure 2: periodicity cell for a mixture giving an upper bound

If  $\varphi(\nu) = c_1|\nu_1| + c_2|\nu_2|$  then such Wulff shape is the rectangle centered in 0 with one vertex in  $(1/(8c_2), 1/(8c_1))$ . A general  $\varphi$  satisfying (1) and (2) corresponds to a convex symmetric set contained in the square of side length  $1/(4\alpha)$  (that is, the Wulff shape corresponding to  $\alpha(|\nu_1| + |\nu_2|)$ ) and containing one of such rectangles for  $c_1$  and  $c_2$  satisfying (2). The envelope of the vertices of such rectangles lies in the curve

$$\frac{1}{|x_1|} + \frac{1}{|x_2|} = 16(\theta\beta + (1 - \theta)\alpha) \quad (3)$$

(see Fig. 3). In terms of such envelope, we can describe the Wulff shapes of  $\varphi$  as follows: if

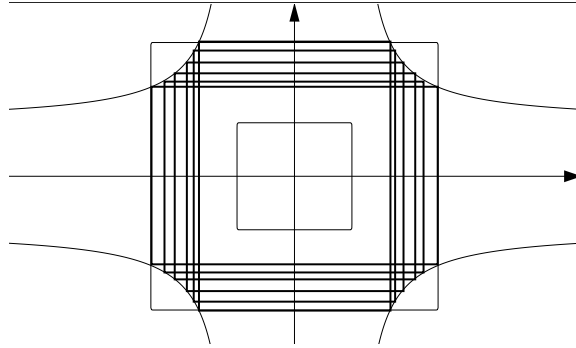


Figure 3: envelope of rectangular Wulff shapes

$\theta \leq 1/2$  then it is any symmetric convex set contained in the square of side length  $1/(4\alpha)$  and intersecting the four portions of the set of points satisfying (3) contained in that square (see Fig. 4(a)); if  $\theta \geq 1/2$  then it is any symmetric convex set contained in the square of side length  $1/(4\alpha)$  and intersecting the four portions of the set of points satisfying (3) with  $|x_1| \geq 1/(8\beta)$  and  $|x_2| \geq 1/(8\beta)$  contained in that square (see Fig. 4(b)). This second condition is automatically satisfied if  $\theta \leq 1/2$ .

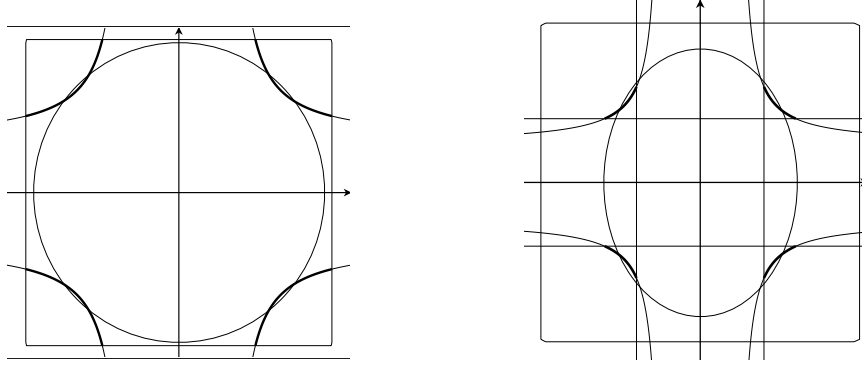


Figure 4: possible Wulff shapes with: (a)  $\theta \leq 1/2$  and (b)  $\theta \geq 1/2$

## 2 Setting of the problem

We consider a discrete system of nearest-neighbour interactions in dimension two with coefficients  $c_{ij} = c_{ji} \geq 0$ ,  $i, j \in \mathbb{Z}^2$ . The corresponding ferromagnetic spin energy is

$$F(u) = \sum_{ij} c_{ij} (u_i - u_j)^2, \quad (4)$$

where  $u : \mathbb{Z}^2 \rightarrow \{-1, 1\}$ ,  $u_i = u(i)$ , and the sum runs over the set of *nearest neighbours* or *bonds* in  $\mathbb{Z}^2$ , which is denoted by

$$\mathcal{Z} = \{(i, j) \in \mathbb{Z}^2 \times \mathbb{Z}^2 : |i - j| = 1\}.$$

Such energies correspond to inhomogeneous surface energies on the continuum [2, 9].

**Definition 1.** Let  $\{c_{ij}\}$  be indices as above with  $\inf_{ij} c_{ij} > 0$  and periodic; i.e., such that there exists  $T \in \mathbb{N}$  such that

$$c_{(i+T)j} = c_{i(j+T)} = c_{ij}.$$

Then, we define the homogenized energy density of  $\{c_{ij}\}$  as the convex positively homogeneous function of degree one  $\varphi : \mathbb{R}^2 \rightarrow [0, +\infty)$  such that for all  $\nu \in S^1$  we have

$$\varphi(\nu) = \lim_{R \rightarrow +\infty} \inf \left\{ \frac{1}{R} \sum_{n=1}^N c_{i_n j_n} : i_N - i_0 = \nu^\perp R + o(R) \right\}. \quad (5)$$

The infimum is taken over all paths of bonds; i.e., pairs  $(i_n, j_n)$  such that the unit segment centred in  $\frac{i_n + j_n}{2}$  and orthogonal to  $i_n - j_n$  has an endpoint in common with the unit segment centred in  $\frac{i_{n-1} + j_{n-1}}{2}$  and orthogonal to  $i_{n-1} - j_{n-1}$ . This is a good definition thanks to [9].

**Remark 2.** The definition above can be interpreted in terms of a passage from a discrete to a continuous description as follows. We consider the scaled energies

$$E_\varepsilon(u) = \frac{1}{8} \sum_{ij} \varepsilon c_{ij}^\varepsilon (u_i - u_j)^2,$$

where  $u : \varepsilon\mathbb{Z}^2 \rightarrow \{-1, 1\}$ , the factor  $1/8$  is a normalization factor, the sum runs over nearest neighbours in  $\varepsilon\mathbb{Z}^2$ , and

$$c_{ij}^\varepsilon = c_{\frac{i}{\varepsilon}, \frac{j}{\varepsilon}}.$$

Upon identifying  $u$  with its piecewise-constant interpolation, we can regard these energies as defined on  $L^1(\mathbb{R}^2)$ . Their  $\Gamma$ -limit in that space is infinite outside  $BV_{\text{loc}}(\mathbb{R}^2, \{\pm 1\})$ , where it has the form

$$F_\varphi(u) = \int_{\partial\{u=1\}} \varphi(\nu) d\mathcal{H}^1$$

with  $\varphi$  as above.

**Periodic mixtures of two types of bonds.** We will consider the case when

$$c_{ij} \in \{\alpha, \beta\} \text{ with } 0 < \alpha < \beta; \quad (6)$$

If we have such coefficients, we define the *volume fraction* of  $\beta$ -bonds as

$$\theta(\{c_{ij}\}) = \frac{1}{4T^2} \# \left\{ (i, j) \in \mathcal{Z} : \frac{i+j}{2} \in [0, T]^2, c_{ij} = \beta \right\}. \quad (7)$$

**Definition 3.** Let  $\theta \in [0, 1]$ . The set of homogenized energy densities of mixtures of  $\alpha$  and  $\beta$  bonds, with volume fraction  $\theta$  (of  $\beta$  bonds) is defined as

$$\begin{aligned} \mathbf{H}_{\alpha, \beta}(\theta) &= \left\{ \varphi : \mathbb{R}^2 \rightarrow [0, +\infty) : \text{there exist } \theta_k \rightarrow \theta, \varphi_k \rightarrow \varphi \text{ and } \{c_{ij}^k\} \right. \\ &\quad \left. \text{with } \theta(\{c_{ij}^k\}) = \theta_k \text{ and } \varphi_k \text{ homogenized energy density of } \{c_{ij}^k\} \right\}. \end{aligned} \quad (8)$$

The following theorem completely characterizes the set  $\mathbf{H}_{\alpha, \beta}(\theta)$ .

**Theorem 4** (optimal bounds). *The elements of the set  $\mathbf{H}_{\alpha, \beta}(\theta)$  are all even convex positively homogeneous functions of degree one  $\varphi : \mathbb{R}^2 \rightarrow [0, +\infty)$  such that*

$$\alpha(|x_1| + |x_2|) \leq \varphi(x_1, x_2) \leq c_1|x_1| + c_2|x_2| \quad (9)$$

for some  $c_1, c_2 \leq \beta$  such that

$$c_1 + c_2 = 2(\theta\beta + (1 - \theta)\alpha). \quad (10)$$

Note that the lower bound for functions in  $\mathbf{H}_{\alpha, \beta}(\theta)$  is independent of  $\beta$ . Note moreover that in the case  $\theta = 1$  we have all functions satisfying the trivial bound

$$\alpha(|x_1| + |x_2|) \leq \varphi(x_1, x_2) \leq \beta(|x_1| + |x_2|). \quad (11)$$

This is due to the fact that in that case by considering  $\theta_k \rightarrow 1$  we allow a vanishing volume fraction of  $\alpha$  bonds, which is nevertheless sufficient to allow for all possible  $\varphi$ .

### 3 Optimality of bounds

We first give two bounds valid for every set of periodic coefficients  $\{c_{ij}\}$ .

**Proposition 5** (bounds by projection). *Let  $\varphi$  be the homogenized energy density of  $\{c_{ij}\}$ ; then we have*

$$\varphi(x) \geq c_1^p |x_1| + c_2^p |x_2|, \quad (12)$$

where

$$c_1^p = \frac{1}{T} \sum_{k=1}^T \min\{c_{ij} : i_2 = j_2 = k\} \quad (13)$$

and

$$c_2^p = \frac{1}{T} \sum_{k=1}^T \min\{c_{ij} : i_1 = j_1 = k\}. \quad (14)$$

*Proof.* The lower bound (12) immediately follows from the definition of  $\varphi$ , by subdividing the contributions of  $c_{i_{n-1}i_n}$  in (5) into those with  $(i_n)_2 = (i_{n-1})_2$  (or equivalently such that  $i_n - i_{n-1} = \pm e_1$ ) and those with  $(i_n)_1 = (i_{n-1})_1$  (or equivalently  $i_n - i_{n-1} = \pm e_2$ , and estimating

$$c_{i_{n-1}i_n} \geq \min\{c_{ij} : i_2 = j_2 = (i_n)_2\}$$

and

$$c_{i_{n-1}i_n} \geq \min\{c_{ij} : i_1 = j_1 = (i_n)_1\},$$

respectively, in the two cases.  $\square$

**Proposition 6** (bounds by averaging). *Let  $\varphi$  be the homogenized energy density of  $\{c_{ij}\}$ ; then we have*

$$\varphi(x) \leq c_1^a |x_1| + c_2^a |x_2|, \quad (15)$$

where  $c_1^a$  is the average over horizontal bonds

$$c_1^a = \frac{1}{T^2} \sum \left\{ c_{ij} : \frac{i+j}{2} \in [0, T)^2, i_2 = j_2 \right\} \quad (16)$$

and  $c_2^a$  is the average over vertical bonds

$$c_2^a = \frac{1}{T^2} \sum \left\{ c_{ij} : \frac{i+j}{2} \in [0, T)^2, i_1 = j_1 \right\}. \quad (17)$$

*Proof.* The proof is obtained by construction of a suitable competitor  $\{i_n, j_n\}$  for the characterization (5) of  $\varphi$ . To that end let  $n_1, n_2 \in \{1, \dots, T\}$  be such that

$$\frac{1}{T} \sum_{k=1}^T c_{(n_1-1,k), (n_1,k)} \leq \frac{1}{T^2} \sum \left\{ c_{ij} : \frac{i+j}{2} \in [0, T)^2, i_2 = j_2 \right\}$$

and

$$\frac{1}{T} \sum_{k=1}^T c_{(k, n_2-1), (k, n_2)} \leq \frac{1}{T^2} \sum \left\{ c_{ij} : \frac{i+j}{2} \in [0, T]^2, i_1 = j_1 \right\}.$$

Up to a translation, we may suppose that  $n_1 = n_2 = 1$ . It is not restrictive to suppose that  $\nu_1 \geq 0$  and  $\nu_2 \geq 0$ . We define  $i_0 = (\lfloor R\nu_2 \rfloor, 0)$  and  $i_N = (0, \lfloor R\nu_1 \rfloor)$ . It suffices then to take in Definition 3 the path of bonds  $\{i_n, j_n\}$  obtained by concatenating the two paths of bonds defined by

$$i_n^1 = (\lfloor R\nu_2 \rfloor - n, 0), \quad j_n^1 = (\lfloor R\nu_2 \rfloor - n, 1), \quad n = 0, \dots, \lfloor R\nu_2 \rfloor - 1$$

and

$$i_n^2 = (0, n), \quad j_n^2 = (1, n), \quad n = 1, \dots, \lfloor R\nu_1 \rfloor.$$

We then have

$$\lim_{R \rightarrow +\infty} \frac{1}{R} \left( \sum_{n=1}^{\lfloor R\nu_2 \rfloor} c_{(n,0), (n,1)} + \sum_{n=1}^{\lfloor R\nu_1 \rfloor} c_{(0,n), (1,n)} \right) = |\nu_2| \frac{1}{T} \sum_{n=1}^T c_{(n,0), (n,1)} + |\nu_1| \frac{1}{T} \sum_{n=1}^T c_{(0,n), (1,n)},$$

and the desired inequality.  $\square$

We now specialize the previous bound to mixtures of two types of bonds. Given  $\{c_{ij}\}$  satisfying (6) we define the *volume fraction of horizontal  $\beta$ -bonds* as

$$\theta_h(\{c_{ij}\}) = \frac{1}{T^2} \# \left\{ (i, j) \in \mathcal{Z} : \frac{i+j}{2} \in [0, T]^2, c_{ij} = \beta, i_2 = j_2 \right\}. \quad (18)$$

and the *volume fraction of vertical  $\beta$ -bonds* as

$$\theta_v(\{c_{ij}\}) = \frac{1}{T^2} \# \left\{ (i, j) \in \mathcal{Z} : \frac{i+j}{2} \in [0, T]^2, c_{ij} = \beta, i_1 = j_1 \right\}. \quad (19)$$

Note that

$$\frac{\theta_h(\{c_{ij}\}) + \theta_v(\{c_{ij}\})}{2} = \theta(\{c_{ij}\}). \quad (20)$$

**Proposition 7.** *Let  $\{c_{ij}\}$  satisfy (6), let  $\theta_h = \theta_h(\{c_{ij}\})$  and  $\theta_v = \theta_v(\{c_{ij}\})$ , and let  $\varphi$  be the homogenized energy density of  $\{c_{ij}\}$ . Then*

$$\varphi(\nu) \leq (\theta_h \beta + (1 - \theta_h) \alpha) |\nu_1| + (\theta_v \beta + (1 - \theta_v) \alpha) |\nu_2| \quad (21)$$

*Proof.* It suffices to rewrite  $c_1^a$  and  $c_2^a$  given by the previous proposition using (18) and (19).  $\square$

The previous proposition, together with (20) and the trivial bound (11) gives the bounds in the statement of Theorem 4. We now prove their optimality. First we deal with a special case, from which the general result will be deduced by approximation.

**Proposition 8.** *Let*

$$\varphi(\nu) = c_1|\nu_1| + c_2|\nu_2|$$

*with  $\alpha \leq c_1, c_2 \leq \beta$  and*

$$c_1 + c_2 \leq 2(\beta\theta + (1 - \theta)\alpha) \quad (22)$$

*for some  $\theta \in (0, 1)$ . Then  $\varphi \in \mathbf{H}_{\alpha, \beta}(\theta)$ .*

*Proof.* The case  $\theta = 1$  is trivial. In the other cases, since the set of  $(c_1, c_2)$  as above coincides with the closure of its interior, by approximation it suffices to consider the case when indeed

$$\alpha < c_1, c_2 < \beta, \quad c_1 + c_2 < 2(\beta\theta + (1 - \theta)\alpha). \quad (23)$$

In particular, we can find  $\theta_1 \in (0, 1)$  and  $\theta_2 \in (0, 1)$  such that  $\theta_1 + \theta_2 = 2\theta$  and

$$c_1 < \beta\theta_1 + (1 - \theta_1)\alpha, \quad c_2 < \beta\theta_2 + (1 - \theta_2)\alpha. \quad (24)$$

We then write

$$c_1 = \beta t_1 + (1 - t_1)\alpha, \quad c_2 = \beta t_2 + (1 - t_2)\alpha. \quad (25)$$

for some  $t_1 < \theta_1$  and  $t_2 < \theta_2$ .

We construct  $\{c_{ij}\}$  with period  $T \in \mathbb{N}$  and with

$$\theta_h(\{c_{ij}\}) = \theta_1, \quad \theta_v(\{c_{ij}\}) = \theta_2$$

by defining separately the horizontal and vertical bonds. Upon an approximation argument we may suppose that  $N_i = t_i T \in \mathbb{N}$ , and that  $T^2 \theta_i \in \mathbb{N}$  for  $i = 1, 2$ . We only describe the construction for the horizontal bonds. We define

$$c_{(j,n),(j+1,n)} = \begin{cases} \beta & \text{if } j = 1, \dots, T \text{ and } n = 1, \dots, N_1 \\ \alpha & \text{if } j = 0 \text{ and } n = N_1 + 1, \dots, T \end{cases}$$

and any choice of  $\alpha$  and  $\beta$  for other indices  $i, j$ , only subject to the total constraint that  $\theta_h(\{c_{ij}\}) = \theta_1$ . With this choice of  $c_{ij}$  we have

$$\min\{c_{ij} : i_2 = j_2 = n\} = \begin{cases} \beta & \text{if } n = 1, \dots, N_1 \\ \alpha & \text{if } n = N_1 + 1, \dots, T \end{cases}$$

The analogous construction for vertical bonds gives

$$\min\{c_{ij} : i_1 = j_1 = n\} = \begin{cases} \beta & \text{if } n = 1, \dots, N_2 \\ \alpha & \text{if } n = N_2 + 1, \dots, T \end{cases}$$

Then, Proposition 5 gives that the homogenized energy density of  $\{c_{ij}\}$  satisfies

$$\varphi(\nu) \geq c_1|\nu_1| + c_2|\nu_2|.$$



To give a lower bound we use the same construction of the proof of Proposition 6, after noticing that the vertical and horizontal paths with  $i_n^1 = (0, n)$ ,  $j_n^1 = (1, n)$  or  $i_n^2 = (n, 0)$ ,  $j_n^2 = (n, 1)$  are such that

$$\frac{1}{T} \sum_{n=1}^T c_{i_n^1, j_n^1} = c_1, \quad \frac{1}{T} \sum_{n=1}^T c_{i_n^2, j_n^2} = c_2.$$

In this way we obtain the estimate

$$\varphi(\nu) \leq c_1 |\nu_1| + c_2 |\nu_2|.$$

and hence the desired equality.  $\square$

*Proof of Theorem 4.* In order to conclude the proof we will use the energy densities obtained in the previous proposition to approximate all  $\varphi$  satisfying the bounds. In order to do this, we note that, thanks to Remark 2 we may use the fact that the class of functionals with integrands in  $\mathbf{H}_{\alpha, \beta}(\theta)$  is closed under  $\Gamma$ -convergence. Hence, it is not restrictive to make some simplifying hypotheses on the function  $\varphi$ .

We may suppose that

$$\alpha(|\nu_1| + |\nu_2|) < \varphi(\nu) < (\beta\theta_1 + (1 - \theta_1)\alpha)|\nu_1| + (\beta\theta_2 + (1 - \theta_2)\alpha)|\nu_2| =: c_1|\nu_1| + c_2|\nu_2|$$

for some  $\theta_1, \theta_2 \in (0, 1)$ , and that

- the set  $\{x : \varphi(x) \leq 1\}$  is a convex symmetric polyhedron with vertices corresponding to integer directions  $\pm\nu^1, \dots, \pm\nu^N$ ; i.e. such that there exist  $r^j \in \mathbb{R}$  such that  $r^j \nu^j \in \mathbb{Z}^d$ .

The surface energy related to such  $\varphi$  can be obtained as a  $\Gamma$ -limit of energies of the form

$$F_\varepsilon(u) = \int_{\partial\{u=1\}} f\left(\frac{x}{\varepsilon}, \nu\right) d\mathcal{H}^1,$$

where  $f(\cdot, \nu)$  is 1-periodic and has the form

$$f(y, \nu) = \begin{cases} \varphi(\nu^k) & \text{if } y \in \{t(\nu^k)^\perp : t \in \mathbb{R}\} + \mathbb{Z}^2, \quad k = 1, \dots, N \\ c_1|\nu_1| + c_2|\nu_2| & \text{otherwise} \end{cases}$$

This can be proved as in [6] or [4], whose construction, where we have  $\beta$  in place of  $c_1|\nu_1| + c_2|\nu_2|$ , works also in this case.

Note that we can rewrite the values

$$\varphi(\nu^k) = (\beta\theta_1^k + (1 - \theta_1^k)\alpha)|\nu_1^k| + (\beta\theta_2^k + (1 - \theta_2^k)\alpha)|\nu_2^k| =: c_1^k|\nu_1^k| + c_2^k|\nu_2^k|$$

with  $c_1^k, c_2^k$  satisfying

$$c_1^k + c_2^k \leq c_1 + c_2.$$

We can therefore consider equivalently

$$f(y, \nu) = \begin{cases} c_1^k |\nu_1| + c_2^k |\nu_2| & \text{if } y \in \{t(\nu^k)^\perp : t \in \mathbb{R}\} + \mathbb{Z}^2, \ k = 1, \dots, N \\ c_1 |\nu_1| + c_2 |\nu_2| & \text{otherwise.} \end{cases}$$

Note in fact that the normal to any  $\partial\{u = 1\}$  will be equal to  $\nu^k$   $\mathcal{H}^1$ -a.e. on  $\{t(\nu^k)^\perp : t \in \mathbb{R}\} + \mathbb{Z}^2$ . This shows that  $f(y, \cdot) \in \mathbf{H}_{\alpha, \beta}(\theta)$  for  $\mathcal{H}^1$ -a.a.  $y$ , and is of the form considered in Proposition 8.

By a further approximation argument the metrics related to such  $f$  can be approximated by a sequence

$$f^\delta(y, \nu) = \begin{cases} c_1^k |\nu_1| + c_2^k |\nu_2| & \text{if } \text{dist}(y, \{t(\nu^k)^\perp : t \in \mathbb{R}\} + \mathbb{Z}^2) \leq \delta \\ & \text{and } \text{dist}(y, \{t(\nu^j)^\perp : t \in \mathbb{R}\} + \mathbb{Z}^2) > \delta, \ j \neq k, \ k = 1, \dots, N \\ c_1 |\nu_1| + c_2 |\nu_2| & \text{otherwise.} \end{cases}$$

By localizing the construction in Proposition 8 we can find  $c_{ij}^{\delta, \eta}$  such that

$$E^\eta(u) = \frac{1}{8} \sum_{ij} \eta c_{ij}^{\delta, \eta} (u_i - u_j)^2 \quad u : \eta \mathbb{Z}^2 \rightarrow \{\pm 1\}$$

$\Gamma$ -converges as  $\eta \rightarrow 0$  to

$$\int_{\partial\{u=1\}} f^\delta(x, \nu) d\mathcal{H}^1$$

Furthermore,  $c_{ij}^{\delta, \eta}$  can be taken periodic of period  $1/\eta$  (which we may suppose being integer) and with horizontal and vertical volume fractions  $\theta_1$  and  $\theta_2$ , respectively.

By a diagonal argument this proves the theorem.  $\square$

## 4 Conclusion and perspectives

The main purpose of this paper has been the construction of discrete microgeometries, that allow the computation of optimal bounds for mixtures of ferromagnetic interaction. To that end we have dealt with the simplest two-dimensional nearest-neighbour setting. There are several extensions of this results: to higher dimension (where the results will be different for length energies and for discrete perimeter functionals); to energies with long-range interactions (for example for nearest and next-to-nearest interactions, where a multi-scale approach can be necessary); to the computations of the *G-closure* of mixtures (i.e., all possible limits of mixtures and not only periodic ones, which, nevertheless, can be reduced to the optimal bounds for periodic mixtures by the *localization principle* of Dal Maso and Kohn), to other lattices (e.g., the triangular lattice), etc.

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